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Symmetries and integrability: Bakirov system revisited

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Abstract

All local generalized symmetries (including x, t-dependent ones) of the Bakirov system are found. In particular, it is shown that its only non-Lie-point local generalized symmetry is the sixth order one found by Bakirov. This result generalizes a similar result of Beukers, Sanders and Wang on x, t-independent symmetries and completes the refutation of the popular conjecture stating that the existence of one non-Lie-point local generalized symmetry for a (1 + 1)-dimensional system of PDEs implies the existence of infinitely many such symmetries.

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1. Introduction

The existence of an infinite number of local generalized symmetries for a given system of PDEs is usually the sign of its linearizability or integrability via the inverse scattering transform (see e.g. [1, 2] and references therein). For a long time it was believed that the existence of one noncontact local generalized symmetry implies the existence of an infinite number of such symmetries for any $(1+1)$-dimensional system of PDEs. By *noncontact* we mean a symmetry that is not equivalent to a Lie point or contact symmetry. In particular, Fokas [3] wrote: '... in all known cases the existence of one generalized symmetry implies the existence of infinitely many'.

Although Sanders and Wang [4] have recently proved this conjecture for x, t-independent symmetries of scalar polynomial $(1+1)$-dimensional evolution equations with a linear leading term, long before this it was understood that the conjecture in question does not hold for the case of systems of PDEs.

Bakirov [5] came up with an example of a (1 + 1)-dimensional evolution system (6). This system was conjectured to have only one x, t-independent noncontact local generalized symmetry $\mathcal{K}$, given below in (7). Using computer algebra, he has shown that this is true for symmetries up to order 53 of this system, but in full generality this result remained unproved for years, and it was proposed as an exercise in Olver's book [1] to find out whether the system in question has noncontact local generalized symmetries other than $\mathcal{K}$. Recently, Beukers *et al* [6] have rigorously proved that $\mathcal{K}$ is the only x, t-independent noncontact local generalized symmetry of the Bakirov system. Their result disproves the above-mentioned conjecture for x, t-independent symmetries.

It is natural to ask what happens to this conjecture if we consider x, t-dependent generalized symmetries as well. The aim of the present paper is to prove that in this case the conjecture also fails. To this end we find all (including x, t-dependent ones) local generalized symmetries of the Bakirov system and show that $\mathcal{K}$ is the only noncontact local generalized symmetry of this system. The main tool we use is the technique of formal symmetries [1, 2, 7–9], combined with certain homogeneity-based arguments (cf e.g. [10, 11]) and investigation of the structure of low order symmetries. We also make substantial use of the results of Beukers, Sanders and Wang [6] on x, t-independent symmetries for the Bakirov system and of Bilge's [12] result on the existence of x, t-independent formal symmetry for this system.

This paper is organized as follows. In section 2 we recall some well known definitions and results concerning the generalized symmetries of evolution equations. In section 3 we present our main result—the complete description of the set of all local generalized symmetries of the Bakirov system. Section 4 contains the detailed proof of this result, and in section 5 we present the discussion.

2. Basic definitions and known results

Given a (1 + 1)-dimensional evolution equation

$$\partial \boldsymbol{u}/\partial t = \boldsymbol{F}(x, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) \qquad n \geqslant 2 \quad \partial \boldsymbol{F}/\partial \boldsymbol{u}_n \neq 0 \tag{1}$$

for an s-component vector function $\boldsymbol{u}$, where $\boldsymbol{u}_l = \partial^l \boldsymbol{u}/\partial x^l$, $l = 0, 1, 2, \ldots, \boldsymbol{u}_0 \equiv \boldsymbol{u}$, consider its (local) *generalized symmetries* [1], i.e., the generalized vector fields $\mathcal{G} = \boldsymbol{G}\,\partial/\partial \boldsymbol{u}$, where $\boldsymbol{G} = \boldsymbol{G}(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)$, $k \in \mathbb{N}$, is such that the evolution equation $\partial \boldsymbol{u}/\partial \tau = \boldsymbol{G}$ is compatible with (1). In what follows we shall often identify the symmetry $\mathcal{G} = \boldsymbol{G}\,\partial/\partial \boldsymbol{u}$ with its *characteristics* $\boldsymbol{G}$.

For any r-component vector function $\vec{H} = \vec{H}(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_q)$ the greatest m such that $\partial \vec{H}/\partial \boldsymbol{u}_m \neq 0$ is called its *order* [2, 8] and is denoted as $m = \mathrm{ord}\vec{H}$. For $\vec{H} = \vec{H}(x, t)$ we assume that $\mathrm{ord}\vec{H} = 0$. We shall call a function $\vec{f}$ of $x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots$ *local* (cf [7, 9]) if it has a finite order.

Denote by $S_{\boldsymbol{F}}^{(k)}$ the space of local generalized symmetries of (1) that are of order not higher than k. In addition, let

$$S_{\boldsymbol{F}} = \bigcup_{j=0}^{\infty} S_{\boldsymbol{F}}^{(j)} \qquad \Theta_{\boldsymbol{F}} = \{\boldsymbol{H}(x, t) \mid \boldsymbol{H}(x, t) \in S_{\boldsymbol{F}}\}$$
$$S_{\boldsymbol{F},k} = S_{\boldsymbol{F}}^{(k)}/S_{\boldsymbol{F}}^{(k-1)} \quad \text{for} \quad k = 1, 2, \ldots \qquad S_{\boldsymbol{F},0} = S_{\boldsymbol{F}}^{(0)}/\Theta_{\boldsymbol{F}}.$$

Finally, let $\mathrm{St}_{\boldsymbol{F}}$ be the set of all time-independent (i.e. *stationary*) local generalized symmetries of (1), that is, $\mathrm{St}_{\boldsymbol{F}} = \{\boldsymbol{G} \in S_{\boldsymbol{F}} \mid \partial \boldsymbol{G}/\partial t = 0\}$.

$S_{\boldsymbol{F}}$ is a Lie algebra with respect to the so-called Lie bracket (see e.g. [1, 9])

$$[\boldsymbol{H}, \boldsymbol{R}] = \boldsymbol{R}_*(\boldsymbol{H}) - \boldsymbol{H}_*(\boldsymbol{R}) = \nabla_{\boldsymbol{H}}(\boldsymbol{R}) - \nabla_{\boldsymbol{R}}(\boldsymbol{H})$$

where for any s-component local vector function $\boldsymbol{Q}$ we have introduced the notation

$$\boldsymbol{Q}_* = \sum_{i=0}^{\mathrm{ord}\boldsymbol{Q}} \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{u}_i} D^i \qquad \nabla_{\boldsymbol{Q}} = \sum_{i=0}^{\infty} D^i(\boldsymbol{Q})\, \frac{\partial}{\partial \boldsymbol{u}_i}.$$

Here, $D = \partial/\partial x + \sum_{i=0}^{\infty} \boldsymbol{u}_{i+1} \partial/\partial \boldsymbol{u}_i$ is the total derivative with respect to x.

Recall (see e.g. [1]) that a local s-component vector function $\boldsymbol{G}$ is a symmetry of (1) if and only if

$$\partial \boldsymbol{G}/\partial t = -[\boldsymbol{F}, \boldsymbol{G}]. \tag{2}$$

It can be shown [1, 7] that (2) implies

$$\partial \boldsymbol{G}_*/\partial t \equiv (\partial \boldsymbol{G}/\partial t)_* = \nabla_{\boldsymbol{G}}(\boldsymbol{F}_*) - \nabla_{\boldsymbol{F}}(\boldsymbol{G}_*) + [\boldsymbol{F}_*, \boldsymbol{G}_*]. \tag{3}$$

Here, $\nabla_{\boldsymbol{F}}(\boldsymbol{G}_*) \equiv \sum_{j=0}^{\mathrm{ord}\boldsymbol{G}} \nabla_{\boldsymbol{F}}\left(\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{u}_j}\right) D^j$ and likewise for $\nabla_{\boldsymbol{G}}(\boldsymbol{F}_*)$, $[\cdot, \cdot]$ stands for the usual commutator of linear differential operators.

Recall (see e.g. [1,2,7,8] for more information) some facts on the formal series in powers of D, i.e., the expressions of the form

$$\mathfrak{H} = \sum_{j=-\infty}^{m} h_j(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots) D^j \tag{4}$$

where h_j are $p \times p$ matrix-valued local functions; in contrast with the above references we let the coefficients of the formal series depend explicitly on time t, but this obviously does not alter the results listed below. Below we shall be interested in the cases when h_j are either scalars or $s \times s$ matrices.

The greatest integer k such that $h_k \neq 0$ is called the degree of formal series $\mathfrak{H}$ (4) and is denoted by $\deg \mathfrak{H}$. A formal series $\mathfrak{H} = \sum_{j=-\infty}^{m} h_j D^j$ of degree m is called *nondegenerate* [8] if its leading coefficient h_m is a nondegenerate matrix.

For any formal series $\mathfrak{H}$ of degree $m \neq 0$ with scalar coefficients there exists a formal series $\mathfrak{H}^{1/m}$ of degree 1 (or -1 for $m < 0$) such that $(\mathfrak{H}^{1/m})^m = \mathfrak{H}$. The formal series $\mathfrak{H}^{1/m}$ is unique up to the multiplication by an mth root of unity [7]. The fractional powers of $\mathfrak{H}$ are defined as $\mathfrak{H}^{l/m} = (\mathfrak{H}^{1/m})^l$ for all integers l, and commute; that is, $[\mathfrak{H}^{p/m}, \mathfrak{H}^{q/m}] = 0$ for all integers p and q, see e.g. [1] for details.

If $\mathfrak{H}$ is a formal series whose coefficients are $s \times s$ diagonal matrices, i.e., $\mathfrak{H} = \mathrm{diag}(\mathfrak{H}_1, \ldots, \mathfrak{H}_s)$, where $\mathfrak{H}_j$ are formal series with *scalar* coefficients, and $\deg \mathfrak{H}_1 = \cdots = \deg \mathfrak{H}_s = m \neq 0$, then we shall, following [2,8], define its mth root as $\mathfrak{H}^{1/m} = \mathrm{diag}(\mathfrak{H}_1^{1/m}, \ldots, \mathfrak{H}_s^{1/m})$. The fractional powers $\mathfrak{H}^{l/m} = (\mathfrak{H}^{1/m})^l$ obviously commute by virtue of commutativity of fractional powers of $\mathfrak{H}_i$, $i = 1, \ldots, s$.

A formal series $\mathfrak{R}$ whose coefficients are $s \times s$ matrices is called a *formal symmetry* (of infinite rank) for (1) if it satisfies the relation (see e.g. [1,2,8])

$$\partial \mathfrak{R}/\partial t + \nabla_{\boldsymbol{F}}(\mathfrak{R}) - [\boldsymbol{F}_*, \mathfrak{R}] = 0. \tag{5}$$

3. Symmetries of the Bakirov system

The Bakirov system has the form [5]

$$\begin{aligned} u_t &= u_4 + v^2 \\ v_t &= \tfrac{1}{5} v_4. \end{aligned} \tag{6}$$

Here, $u_j = \partial^j u/\partial x^j$, $v_j = \partial^j v/\partial x^j$. We shall also employ the notation $\boldsymbol{u}_j = (u_j, v_j)^T$, $\boldsymbol{u} = \boldsymbol{u}_0 = (u, v)^T$, where superscript '$T$' stands for the matrix transposition. To refer to sets of symmetries of the Bakirov system, we shall use the subscript 'Bak' instead of $\boldsymbol{F}$, i.e., S_{Bak} will denote the Lie algebra of all generalized symmetries of (6), etc. From now on $\boldsymbol{F}$ will stand for the right-hand side of the Bakirov system, that is, $(u_4 + v^2, v_4/5)^T$.

The straightforward computation of local generalized symmetries of orders $0, \ldots, 6$ of the Bakirov system shows that any of its symmetries of order not higher than six is a linear combination of the following symmetries:

$$\begin{aligned} &\mathcal{X} = u_1\,\partial/\partial u + v_1\,\partial/\partial v && \mathcal{T} = (u_4 + v^2)\,\partial/\partial u + \tfrac{1}{5}v_4\,\partial/\partial v \\ &\mathcal{D}_0 = 2u\,\partial/\partial u + v\,\partial/\partial v && \mathcal{D} = 4t\mathcal{T} + x\mathcal{X} + 2v\,\partial/\partial v \\ &\mathcal{W}_\alpha = \alpha(x,t)\,\partial/\partial u && \\ &\mathcal{K} = (u_6 + \tfrac{5}{11}(5vv_2 + 4v_1^2))\,\partial/\partial u + \tfrac{1}{11}v_6\,\partial/\partial v && \end{aligned} \tag{7}$$

where $\alpha(x,t)$ is any sufficiently smooth solution of the equation $\alpha_t = \alpha_4$. Note that a highly reduced system of determining equations for these symmetries was obtained using M Marvan's program *Jet*, version 4.3 for Maple V Release 4.

For ease of reading we have presented in (7) the complete formulae for symmetries and not just for their characteristics.

Our main result is the following theorem.

Theorem 1. *Any local generalized symmetry of the Bakirov system is a linear combination of the symmetries from list (7).*

All local generalized symmetries from (7) except for $\mathcal{K}$ are equivalent to Lie point ones, and thus theorem 1 implies that $\mathcal{K}$ is the only noncontact local generalized symmetry of the Bakirov system.

4. The structure of the symmetries of the Bakirov system

4.1. On time dependence of symmetries

Let $\boldsymbol{G}$ be a local generalized symmetry of order $k \geqslant 0$ for (6). Solving the equations obtained by equating to zero the coefficients at D^{k+4} and D^{k+3} in (3), it is easy to show (cf e.g. [1, 9]) that

$$\partial \boldsymbol{G}/\partial \boldsymbol{u}_k = c_k(t) \tag{8}$$

where $c_k(t)$ is a diagonal 2×2 matrix-valued function of t.

In what follows we assume without loss of generality that any symmetry $\boldsymbol{G} \in S_{\mathrm{Bak},k}$, $k \geqslant 0$, vanishes provided the relevant function $c_k(t)$ is identically equal to zero.

Let Γ denote the set of symmetries $\mathcal{W}_\alpha$ for all (smooth) solutions $\alpha(x,t)$ of the equation $\alpha_t = \alpha_4$. Considering a symmetry $\boldsymbol{G} \in S_{\mathrm{Bak}}$ of order k and successively solving the determining equations for $\partial\boldsymbol{G}/\partial\boldsymbol{u}_j$ with $j = k-1, k-2, \ldots$ that follow from (3), and using (8), we readily see that $\boldsymbol{G}$ is linear in u_j for $j = 0, \ldots, k$ and, what is more, G_2 is independent of u_j for all j, whence $[\boldsymbol{G}, \boldsymbol{H}] \in \Gamma$ for any $\boldsymbol{H} \in \Gamma$. Thus, Γ is an ideal in the Lie algebra S_{Bak}. Therefore, the quotient space $S'_{\mathrm{Bak}} = S_{\mathrm{Bak}}/\Gamma$ is a Lie subalgebra in S_{Bak}. In particular, $[\boldsymbol{F}, \boldsymbol{G}] = -\partial\boldsymbol{G}/\partial t \in S'_{\mathrm{Bak}}$ and $[\boldsymbol{u}_1, \boldsymbol{G}] = -\partial\boldsymbol{G}/\partial x \in S'_{\mathrm{Bak}}$ for any $\boldsymbol{G} \in S'_{\mathrm{Bak}}$.

Moreover, successively equating to zero the coefficients at D^{k+2}, D^{k+1}, D^k in (3) and analysing thus obtained equations, it is easy to show that $\partial^2\boldsymbol{G}/\partial\boldsymbol{u}_j\partial x = 0$ for $j = k-2$ and $j = k-1$, and

$$\frac{\partial^2 \boldsymbol{G}}{\partial \boldsymbol{u}_{k-3}\,\partial x} = \frac{1}{4}\dot{c}_k(t)\Psi \qquad \Psi = \mathrm{diag}\,(1,5). \tag{9}$$

Lemma 1. *All symmetries from the space S'_{Bak} are polynomial in time.*

Proof. For the symmetries of orders $0, \ldots, 6$ this is immediate from (7). Now assume our result to be already proved for the symmetries of order lower than k and consider a symmetry

$\boldsymbol{G} \in S'_{\mathrm{Bak},k} \equiv S'_{\mathrm{Bak}} \cap S_{\mathrm{Bak},k}$, $k \geqslant 7$. Obviously, $\tilde{\boldsymbol{G}} = \partial^r \boldsymbol{G}/\partial x^r \in S'_{\mathrm{Bak}}$. Successively using (9), we see that $\mathrm{ord}\tilde{\boldsymbol{G}} \leqslant k - 3r$ and

$$\frac{\partial \tilde{\boldsymbol{G}}}{\partial \boldsymbol{u}_{k-3r}} = \frac{\partial^r c_k(t)}{\partial t^r}(\Psi/4)^r. \qquad (10)$$

In particular, for $r = [k/3]$ we have $\mathrm{ord}\tilde{\boldsymbol{G}} \leqslant 2$. It is clear from (7) that all symmetries from $S^{(2)}_{\mathrm{Bak}} \cap S'_{\mathrm{Bak}}$ are time independent. Hence, by virtue of (10) the function $c_k(t) = \partial \boldsymbol{G}/\partial \boldsymbol{u}_k$ satisfies the equation $\partial^q c_k(t)/\partial t^q = 0$ with $q = [k/3] + 1$. Therefore, the order of symmetry $\partial^q \boldsymbol{G}/\partial t^q \in S'_{\mathrm{Bak}}$ is not higher than $k - 1$, and by an inductive hypothesis this symmetry is a polynomial in t, whence it is immediate that so is $\boldsymbol{G}$ itself, and the result follows. □

The Bakirov system is invariant under the scaling symmetry $\mathcal{D} \equiv \boldsymbol{D}\partial/\partial\boldsymbol{u}$. Hence, if a symmetry $\mathcal{Q} = \boldsymbol{Q}\partial/\partial\boldsymbol{u}$ contains the terms of weight γ (with respect to the weighting induced by $\mathcal{D}$, cf [6]), there obviously exists a homogeneous symmetry $\tilde{\mathcal{Q}} = \tilde{\boldsymbol{Q}}\partial/\partial\boldsymbol{u}$ of the same weight γ. We shall write this as $\mathrm{wt}(\tilde{\mathcal{Q}}) = \gamma$. Note that we have $[\boldsymbol{D}, \tilde{\boldsymbol{Q}}] = \gamma\tilde{\boldsymbol{Q}}$.

Next, let $\boldsymbol{G} \in S'_{\mathrm{Bak},k}$, $k \geqslant 1$, be a polynomial in t of degree m; that is, $\boldsymbol{G} = \sum_{j=0}^{m} t^j \boldsymbol{g}_j(x, \boldsymbol{u}, \ldots, \boldsymbol{u}_k)$, $\boldsymbol{g}_m \neq 0$. It is clear that $\partial \boldsymbol{G}/\partial \boldsymbol{u}_k = c_k(t)$ is also a polynomial in t of degree $m' \leqslant m$, i.e., $c_k(t) = \sum_{j=0}^{m'} t^j c_{k,j}$, where $c_{k,m'} \neq 0$.

Consider $\tilde{\boldsymbol{G}} = \partial^{m'}\boldsymbol{G}/\partial t^{m'} \in S^{(k)}_{\mathrm{Bak}} \cap S'_{\mathrm{Bak}}$. Since $\partial\tilde{\boldsymbol{G}}/\partial\boldsymbol{u}_k$ is a nonzero constant matrix, it is immediate that $\tilde{\mathcal{G}} = \tilde{\boldsymbol{G}}\partial/\partial\boldsymbol{u}$ contains the terms of the weight k. In turn, this implies that there exists a time-independent symmetry $\boldsymbol{P} \in S'_{\mathrm{Bak}}$ of order k such that $\mathcal{P} = \boldsymbol{P}\partial/\partial\boldsymbol{u}$ is of weight k. Indeed, let $\tilde{\mathcal{P}} = \tilde{\boldsymbol{P}}\partial/\partial\boldsymbol{u}$ denote the homogeneous component of $\tilde{\mathcal{G}}$ of weight k (i.e. the set of all terms of weight k in $\tilde{\mathcal{G}}$), and let $\boldsymbol{P}$ be the projection of $\tilde{\boldsymbol{P}}$ on S'_{Bak}. We readily see that $\mathcal{P} = \boldsymbol{P}\partial/\partial\boldsymbol{u}$ is also homogeneous of weight k, because $\mathcal{D} \in S'_{\mathrm{Bak}}$. Obviously, $\boldsymbol{P}$ has order k and its leading term $\partial\boldsymbol{P}/\partial\boldsymbol{u}_k$ is a constant matrix. As $\boldsymbol{P}$ is a symmetry of the Bakirov system, $\partial\boldsymbol{P}/\partial t = -[\boldsymbol{F}, \boldsymbol{P}] \in S'_{\mathrm{Bak}}$, and the symmetry $\partial\mathcal{P}/\partial t = (\partial\boldsymbol{P}/\partial t)\partial/\partial\boldsymbol{u}$ is homogeneous of weight $k+4$. It is straightforward to verify that $\mathrm{ord}\partial\boldsymbol{P}/\partial t \leqslant k-1$. Hence, if S'_{Bak} contains no homogeneous symmetries $\boldsymbol{Q}$ such that $\mathrm{wt}(\boldsymbol{Q}\partial/\partial\boldsymbol{u}) = k+4$ and $\mathrm{ord}\boldsymbol{Q} \leqslant k-1$, then $\partial\boldsymbol{P}/\partial t = 0$, i.e. $\boldsymbol{P}$ is time independent.

However, the existence of the scaling symmetry $\mathcal{D}$ for the Bakirov system readily implies the existence of a basis in S'_{Bak} made of homogeneous symmetries. As all symmetries in S'_{Bak} are polynomial in t by virtue of the above lemma, so are their leading terms, and thus for any homogeneous symmetry $\mathcal{B} = \boldsymbol{B}\partial/\partial\boldsymbol{u}$, $\boldsymbol{B} \in S'_{\mathrm{Bak}}$, $b \equiv \mathrm{ord}\boldsymbol{B} \geqslant 0$, we have $\partial\boldsymbol{B}/\partial\boldsymbol{u}_b = t^r c_b$ for some $r \geqslant 0$, where c_b is a constant 2×2 diagonal matrix. This observation along with the homogeneity of $\mathcal{B} \equiv \boldsymbol{B}\partial/\partial\boldsymbol{u}$ readily implies that $\mathrm{wt}(\mathcal{B}) = b - 4r \leqslant b$. Hence, for $k \geqslant 1$ the set S'_{Bak} indeed does not contain homogeneous symmetries $\boldsymbol{Q}$ such that $\mathrm{wt}(\boldsymbol{Q}\partial/\partial\boldsymbol{u}) = k + 4$ and $\mathrm{ord}\boldsymbol{Q} \leqslant k - 1$, so $\partial\boldsymbol{P}/\partial t = 0$, and the result follows.

Summing up the above arguments, we conclude that the necessary condition for the existence of a polynomial-in-time symmetry $\boldsymbol{G} \in S'_{\mathrm{Bak}}$ of order $k \geqslant 1$ is the existence of a time-independent symmetry of the same order from S'_{Bak}. Moreover, by lemma 1 all symmetries from S'_{Bak} are polynomial in t. Hence, the absence of time-independent local generalized symmetries of order higher than p for some $p \geqslant 1$ for the Bakirov system immediately implies the absence of any *time-dependent* local generalized symmetries of order higher than p.

The above reasoning can be restricted to x-independent symmetries. Hence, the absence of t, x-independent symmetries of order higher than six for the Bakirov system, proved by Beukers, Sanders and Wang [6], implies that this system has no time-dependent local generalized symmetries of order higher than six that are independent of x.

4.2. The structure of time-independent symmetries

By proposition 3.1 from [8] there exists a unique formal series $\mathfrak{T} = E + \sum_{j=-\infty}^{-1} \Omega_j D^j$ such that all coefficients of the formal series $\mathfrak{V} = \mathfrak{T}\boldsymbol{F}_*\mathfrak{T}^{-1} + D_t(\mathfrak{T})\mathfrak{T}^{-1}$ are diagonal 2×2 matrices. Here, Ω_j are x, t-independent 2×2 matrix-valued local functions with zero diagonal entries, and E is 2×2 unit matrix.

Consider the formal series $\mathfrak{P} = \mathfrak{T}\boldsymbol{G}_*\mathfrak{T}^{-1}$ for $\boldsymbol{G} \in \mathrm{St}_{\mathrm{Bak}}$. Since $\partial\boldsymbol{G}/\partial t = 0$, we readily obtain from (3) that $[\nabla_{\boldsymbol{F}} - \mathfrak{V}, \mathfrak{P}] = \mathfrak{T}\nabla_{\boldsymbol{G}}(\boldsymbol{F}_*)\mathfrak{T}^{-1}$. It is easy to see that for the Bakirov system we have $\deg \nabla_{\boldsymbol{G}}(\boldsymbol{F}_*) = \deg \mathfrak{T}\nabla_{\boldsymbol{G}}(\boldsymbol{F}_*)\mathfrak{T}^{-1} \leqslant 0$, and hence $\deg [\nabla_{\boldsymbol{F}} - \mathfrak{V}, \mathfrak{P}] \leqslant 0$.

As was shown by Bilge [12], the Bakirov system has a nondegenerate formal symmetry $\mathfrak{L}$ of degree 2 and of infinite rank with x, t-independent coefficients. Since $\mathfrak{L}$ by definition satisfies the equation $[\nabla_{\boldsymbol{F}} - \boldsymbol{F}_*, \mathfrak{L}] = 0$, $\mathfrak{L}' = \mathfrak{T}\mathfrak{L}\mathfrak{T}^{-1}$ satisfies $\left[\nabla_{\boldsymbol{F}} - \mathfrak{V}, \mathfrak{L}'\right] = 0$ and, by virtue of the results of [8], the coefficients of $\mathfrak{L}'$ are diagonal 2×2 matrices.

Since $\deg [\nabla_{\boldsymbol{F}} - \mathfrak{V}, \mathfrak{P}] \leqslant 0$, in analogy with lemma 9 from [9] we can represent $\mathfrak{P}$ in the form $\mathfrak{P} = \sum_{j=0}^{k} \alpha_j (\mathfrak{L}')^{j/2} + \mathfrak{B}$, where α_j are constant diagonal 2×2 matrices and $\mathfrak{B}$ is some formal series with time-independent coefficients, $\deg \mathfrak{B} < 0$. Hence

$$\boldsymbol{G}_* = \mathfrak{T}^{-1}\left(\sum_{j=0}^{k} \alpha_j (\mathfrak{L}')^{j/2} + \mathfrak{B}\right)\mathfrak{T}.$$

We have $\partial\mathfrak{T}/\partial x = 0$ and $\partial\mathfrak{L}'/\partial x = 0$. Therefore, $\partial\boldsymbol{G}_*/\partial x = \mathfrak{T}^{-1}\partial\mathfrak{B}/\partial x\mathfrak{T}$. But it is clear that provided $\partial\boldsymbol{G}_*/\partial x \neq 0$ we have $\deg \partial\boldsymbol{G}_*/\partial x \geqslant 0$, while $\deg \mathfrak{T}^{-1}\partial\mathfrak{B}/\partial x\mathfrak{T} < 0$. This contradiction readily implies that $\partial\boldsymbol{G}_*/\partial x = 0$.

Thus, any symmetry $\boldsymbol{G} \in \mathrm{St}_{\mathrm{Bak}}$, $k \equiv \mathrm{ord}\boldsymbol{G} \geqslant 0$, can be represented in the form

$$\boldsymbol{G} = \boldsymbol{G}_0(\boldsymbol{u}, \ldots, \boldsymbol{u}_k) + \boldsymbol{Y}(x). \tag{11}$$

It is obvious that $\partial\boldsymbol{Y}/\partial x = \partial\boldsymbol{G}/\partial x \in \mathrm{St}_{\mathrm{Bak}}$. Hence, by (7) the components of $\boldsymbol{Y}$ have the form

$$Y_1(x) = \sum_{j=1}^{4} c_j \frac{x^j}{j!} \qquad Y_2(x) = 0$$

where c_j are some constants. We omit the integration constants in Y_i, because they can always be included in $\boldsymbol{G}_0$.

Let us prove that $c_4 = 0$ for any $\boldsymbol{G} \in \mathrm{St}_{\mathrm{Bak}}$. Since $(x^j, 0)^T$ for $j = 0, 1, 2, 3$ belong to $\mathrm{St}_{\mathrm{Bak}}$, we only have to show that the Bakirov system has no time-independent symmetries of the form $\boldsymbol{G} = \boldsymbol{G}_0(\boldsymbol{u}, \ldots, \boldsymbol{u}_k) + \boldsymbol{Y}(x)$ with $\boldsymbol{Y}(x) = (cx^4/4!, 0)^T$, $c = \mathrm{const}$, $c \neq 0$. By definition any such symmetry satisfies the equation $[\boldsymbol{F}, \boldsymbol{G}] = 0$, which is equivalent to the following one:

$$[\boldsymbol{F}, \boldsymbol{G}_0] = -4(c, 0)^T \equiv -\boldsymbol{H}. \tag{12}$$

We have $\boldsymbol{H} \in \mathrm{St}_{\mathrm{Bak}}$, so (12) implies the existence of time-dependent (but x-independent) symmetry $\boldsymbol{G}_0 + t\boldsymbol{H}$ of the Bakirov system (6).

However, it is clear from the above that (6) has no polynomial-in-time and x-independent local generalized symmetries of order higher than six. Hence, as $\boldsymbol{G}_0 + t\boldsymbol{H}$ is x-independent and linear in time t by construction, we have $\mathrm{ord}\boldsymbol{G}_0 \leqslant \mathrm{ord}(\boldsymbol{G}_0 + t\boldsymbol{H}) \leqslant 6$. But it is immediate from (7) that the Bakirov system has no local generalized symmetries of the form $\boldsymbol{G}_0 + t\boldsymbol{H}$ with $\partial\boldsymbol{G}_0/\partial x = 0$ and $\boldsymbol{H} \neq 0$ of orders $0, \ldots, 6$, so it has no such symmetries (of any order) at all, and the result follows.

Thus, we have proved the following lemma.

Lemma 2. *Any symmetry* $\boldsymbol{G} \in \mathrm{St}_{\mathrm{Bak}}$ *is a linear combination of* x, t*-independent symmetries and of the symmetries* $\boldsymbol{B}_j = (x^j, 0)^T$, $j = 1, 2, 3$.

Lemma 2, in combination with the result of Beukers *et al* [6] on the absence of x, t-independent generalized symmetries of order higher than six for the Bakirov system, immediately implies that this system has no time-independent local generalized symmetries of order higher than six, including x-dependent ones.

In turn, by virtue of the results of the previous section this implies that the Bakirov system has no polynomial-in-time local generalized symmetries of order higher than six. Finally, by lemma 1 all symmetries of this system from the space S'_{Bak} are exhausted by polynomials in time t, and hence the Bakirov system has no local generalized symmetries of order higher than six at all. On the other hand, all its local generalized symmetries up to sixth order can be found by straightforward computation and are listed in (7). Thus, any local generalized symmetry of the Bakirov system is a linear combination of symmetries from the set (7), and theorem 1 is proved.

5. Conclusions and discussion

We have shown above that all local generalized symmetries of the Bakirov system (6) are exhausted by those from (7). In particular, $\mathcal{K}$ from (7) turned out to be the only noncontact local generalized symmetry of this system. Our result generalizes a similar statement of Beukers, Sanders and Wang [6] concerning x, t-independent symmetries, and gives the final negative answer to the question of, posed by Olver in [1], whether this system has local generalized symmetries other than $\mathcal{K}$ that are not equivalent to Lie point or contact symmetries. What is more, our result completes the refutation of the conjecture stating that if a $(1+1)$-dimensional system of PDEs has one generalized symmetry that is not equivalent to a Lie point or contact one, then it has infinitely many such symmetries, see the introduction for details.

As a final remark, let us mention that Fokas [13] suggested a modified version of the above-mentioned conjecture for evolution systems stating that if a $(1 + 1)$-dimensional s-component evolution system has s time-independent non-Lie-point generalized symmetries, then it has infinitely many such symmetries. Sanders and van der Kamp [14] have disproved this conjecture for x-independent symmetries by exhibiting a wide class of two-component evolution systems with only two x, t-independent non-Lie-point local generalized symmetries. The methods of the present paper are applicable to the systems from [14] and enable one to find all their local generalized symmetries, including x, t-dependent ones. In particular, it is possible to complete the refutation of the modified Fokas conjecture for x-dependent (and for x, t-dependent) symmetries by proving that the systems from [14] have only a finite number of non-Lie-point local generalized symmetries, including x, t-dependent ones. We intend to present the detailed proof of this result elsewhere.

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